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CONCEPTS, ORIGINS, AND USE  
OF LINEAR PROGRAMMING

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SUMMARY

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This paper discusses the concept, origins, and some of  
the applications of linear programming. ( )  
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## CONCEPTS, ORIGINS, AND USE OF LINEAR PROGRAMMING

George B. Dantzig

### 1. CONCEPTS

Too often linear programming is introduced by citing a number of miscellaneous examples that appear to have little in common except that their mathematical description requires a solution of a system of linear inequalities that minimizes (or maximizes) a linear form. Conceptually linear programming is concerned with building a model for describing the interrelations of the components of a system. It has a certain philosophy or approach to model building that has application to a broad class of decision problems. Since this important aspect of linear programming has not been receiving enough emphasis, we shall begin with a review.

The first step in building a model consists in regarding a system under design as composed of a number of elementary functions that are called "activities" (von Neumann<sup>1</sup> used the term "process" while T. C. Koopmans<sup>2</sup> coined the term "Activity Analysis" to describe this approach). The multitude of different type activities in which a system can engage constitutes its technology; these are the representative building blocks that can be recombined in varying amounts for form a new system. The

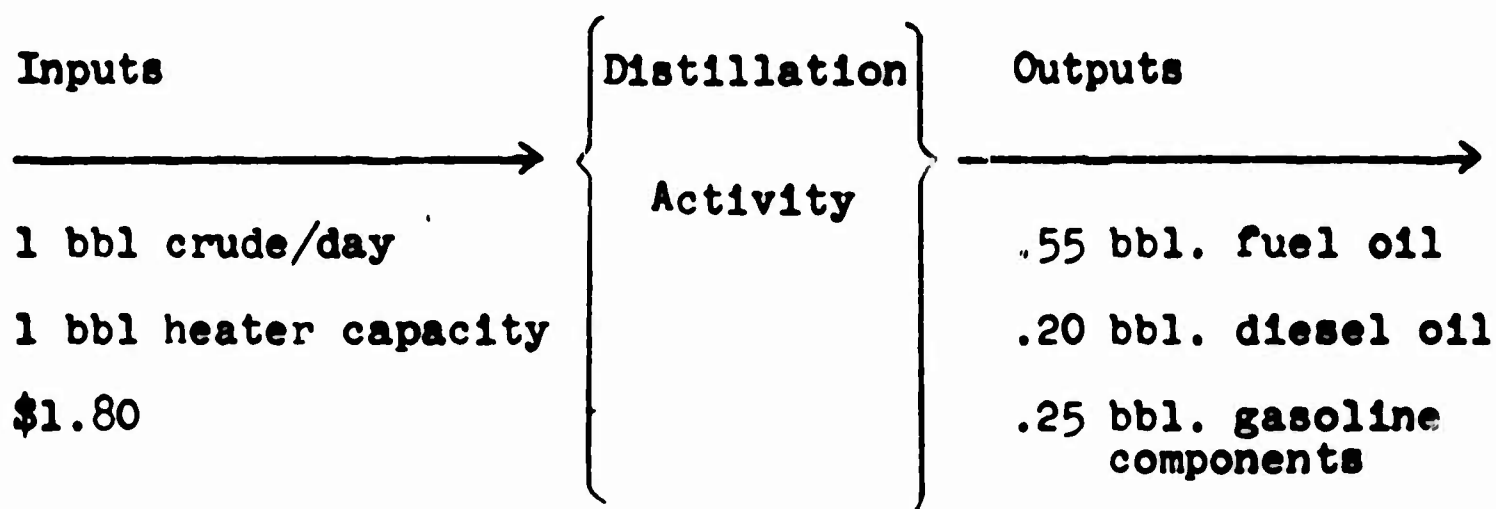
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<sup>1</sup>J. von Neumann, "A Model of General Economic Equilibrium," Review of Economic Studies, Vol. XIII (1), No. 33, 1945-46, pp. 1-9.

<sup>2</sup>"Activity Analysis of Production and Allocation," T. C. Koopmans editor, Wiley and Sons publisher, 1951. The formulation of the linear programming model can be found in Chapter I, "The Programming of Interdependent Activities—Mathematical Discussion" by George B. Dantzig.

quantities of different types of activities to be performed by the new system is called its program. It becomes a feasible program, if the structure reared by the blocks is mutually self-supporting and consistent with the inputs available from outside the system and the outputs required to be produced by the system.

As for the building blocks themselves, each activity is viewed as a kind of "black box" in which the quantities of various items, such as supply, people, money, and machinery, that flow into it are transformed into different quantities of items that flow out. The black box, of course, is a myth. What we mean is that, given the stated input flows, we have the technical know-how to combine them to produce the stated output flows. For example, in the refining of crude oil, a simple distillation activity of a certain type crude might be characterized by



It seems quite reasonable to suppose that if twice (or any positive multiple) of the input flows were made available that the same technical know-how could be used to produce

corresponding multiple of output flows. Thus distilling twice as many barrels of crude per day would use twice as many barrels of crude per day, would require twice the distillation capacity, would cost twice as much, and would produce twice as many barrels of fuel, diesel, and the lighter type oils (which are used as components in the blending of gasoline stocks). It is this proportionality aspect more than any other that characterizes the linear programming model. Assuming its general truth within a model, it is natural to consider all activities that can be generated as a multiple of a fixed activity as forming an activity type; each activity within the type is characterized by the multiple of the fixed activity called the "activity level."

Associated with each item that can flow into or out of each activity is a material balance equation. To be precise, for each item, it is required that the total amount on hand equal the amount flowing into the various activities minus the amount flowing out.

Finally, one of the items of the system is regarded as precious in the sense that the total amount of it produced by the system measures the payoff. The programming problem consists in determining values for the activity levels which are positive or zero such that the flows of each item (for these activity levels) satisfies the material balance equations and such that the value of the payoff is maximum.

These steps, if followed, lead to a well-defined mathematical model of the system, called the linear programming model.

Indeed, let the subscript  $j = 1, 2, \dots, n$  denote the  $j^{\text{th}}$  type activity and  $x_j$  its quantity; let the subscript  $i$  denote the  $i^{\text{th}}$  type item and  $a_{ij}$  denote the input of the  $i^{\text{th}}$  type item into the  $j^{\text{th}}$  type activity per unit quantity of activity ( $a_{ij}$ , if negative, denotes output instead of input); let  $b_i$  denote the given available input to the system from exogeneous sources ( $b_i$ , if negative, denotes the required outputs from the system). The material balance equations then become

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$(1) \quad \dots \dots \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

where the term  $a_{ij}x_j$  denotes the total flow of the  $i^{\text{th}}$  item into the  $j^{\text{th}}$  type activity if its activity level is  $x_j$ . For the precious item let  $c_j$  denote the "cost" per unit level of  $j^{\text{th}}$  activity; then

$$(2) \quad c_1x_1 + c_2x_2 + \dots + c_nx_n = z$$

measures the total system costs ( $-z$  measures total payoff).

The linear programming problem then consists in choosing numbers

$$(3) \quad x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

satisfying (1), such that the value of  $z$  given by (2) is minimum.

## 2. ORIGINS

### ECONOMIC ORIGINS OF LINEAR PROGRAMMING

Linear programming is an anachronism. Logically it should have begun around 1760 when economists first began to describe economic systems in mathematical terms. Thus a crude example of a linear programming model is found in the "Tableau Economique" of Quesnay, who attempted to interrelate the roles of the landlord, the peasant and the artisan. During the next 175 years there was, however, little in the way of exploitation of a linear-type model although it did appear as part of the Walrasian system in the 1870's. For the most part, mathematical economists made use of general functions whose parameters were as a rule unspecified. In their defense, it should be remembered that very little in the way of facts were available on income, quantities of production, investments, savings, distribution, etc., so that the purpose of the mathematical equations was to describe in a qualitative rather than a quantitative way the assumed interrelations within a system while the manipulation of equations corresponded to logical deductions from the assumptions.

The first major impetus to the construction of practical mathematical models to describe an economy, came about in the 1930's as a result of the great depression for, with the advent of the "New Deal," there was a serious attempt on the part of the government to support certain activities which, it was hoped, would speed recovery.



Wassily Leontief, professor at Harvard, brought out his book "The Structure of the American Economy" during this period. This marked a sharp departure from earlier model building because a large number of industries (or sectors) were inter-related in a single model that permitted estimation of the parameters in a practical and quantitative way. Each "sector" represented a number of related functions that were lumped together and called an industry, such as steel. By estimating the dollar sales of each sector to every other sector it was possible to develop a so-called input-output model. This was done by assuming the dollar input of products into sector A from B was proportional to the dollar output of sector A. The scope, accuracy, and area of application of Leontief-type models were greatly extended by the Bureau of Labor Statistics in the years that followed. It was this work that was generalized in the post war period into a form suitable for dynamic Air Force applications.

About the same time that Leontief was producing his model, John von Neumann, world-famous mathematician, published his "Model of Economic Equilibrium," (1937). His model (like the Leontief model) was a linear programming model. It was more general, as it allowed for alternative activities. It was completely general except it did not possess a general objective form. His purpose was limited to proving a relation between the expansion rate and the interest rate on money for a constantly expanding economy. As far as influence is concerned, this paper, like many others, had for purpose, an interesting mathematical

theorem. There appears to have been little interest in the model itself among economists, for, according to T. C. Koopmans, "To many economists, the term linearity is associated with narrowness, restrictiveness, and inflexibility of hypothesis."<sup>1</sup>

However, it was the broad applicability of the linear programming approach to military planning that stimulated economists during the post war period to recognize the potentialities of this type model for studying economic problems.

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<sup>1</sup>"Introduction," Activity Analysis of Production and Allocation, T. C. Koopmans (Editor), page 6.

## MILITARY ORIGINS OF PROGRAM PLANNING

The following quotation<sup>1</sup> from a paper by Wood and Geisler traces developments in the military area through World War II:

"It was once possible for a supreme commander to plan operations personally. As the planning problem expanded in space, time, and general complexity, however, the inherent limitations in the capacity of any one man were encountered. Military histories are filled with instances of commanders who failed because they bogged down in details, not because they could not eventually have mastered the details, but because they could not master all the relevant details in the time available for decision.

"Gradually, as planning problems became more complex, the supreme commander came to be surrounded with a general staff of specialists, which supplemented the chief in making decisions. The existence of a general staff permitted the subdivision of the planning process and the assignment of experts to handle each part. The function of the chief then became one of selecting objectives, coordinating, planning, and resolving conflicts between staff sections."

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<sup>1</sup>The quotation that follows is taken from a paper by Marshall K. Wood and Murray A. Geisler, "Development of Dynamic Models for Program Planning," Chapter XII, Activity Analysis of Production and Allocation, pp. 189-192.

In the U.S. Air Force, for example, during the war a group under the guidance of Professor E. P. Learned of Harvard Business School evolved a scheme which was a program for programming by means of which the work of many staff and command agencies participating in the process was carefully scheduled. In order to obtain consistent programming the ordering of the steps in the schedule were so arranged that the flow of information from echelon to echelon was only in one direction. Even with the most careful scheduling, it took about seven months to complete the process.

After the war the Air Force consolidated the statistical control, programming monitoring, and budgeting functions under the Air Force Comptroller—General E. W. Rawlings. It became clear to members of this organization that coordinating quickly and well the energies of whole nations in a total war required scientific programming techniques. Undoubtedly this need had occurred many times in the past but this time the means of accomplishment were at hand—these were the electronic computer developments and a new way to describe the interrelations of a complex organization so as to permit computation on electronic computers. Here, of course, I am referring to linear programming.

Intensive work began in June 1947 in a group that later (October 1948) was given the official title of Project "SCOOP" (Scientific Computation of Optimum Programs).

While the initial proposal was to use a linear programming model to develop Air Force programs, it was recognized at an

early date that even the most optimistic estimates of the efficiency of future computing procedures and facilities would not be powerful enough to prepare detailed Air Force programs. Accordingly, in the Spring of 1948, there was a second SCOOP proposal that special linear programming models, called triangular models, be developed whose computational solution would parallel the stepwise staff procedure. Since 1948 the Air Staff has been making more and more active use of mechanically computed programs. The triangular models are used by them for detailed programs, while the general linear programming models are used in certain areas.

### 3. APPLICATIONS

It is of course impossible in a brief talk to do more than give a short list of military and industrial applications and to illustrate briefly. Military applications include:

1. Optimal routing for air transport
2. Time phased distribution of supply from factories and depots to bases
3. Allocation of electronic equipment to naval vessels
4. Production smoothing problems
5. Contract award problems
6. Communications system design and message routing
7. Personnel assignment problems
8. Maximal flow in transportation networks
9. Minimal tanker fleets to meet a fixed schedule

Some of these which I have cited represent areas where the linear programming tool has been proposed or has been used from time to time. Others, like the production smoothing, contract awards, and supply distribution, have found frequent application.

In the spring of 1951, four years after the birth of linear programming, Charnes and Cooper at Carnegie began to pioneer industrial applications. With Bob Mellon of Gulf Oil Company, they were the first to develop the application of linear programming to refinery problems. This has since proved to be the most fruitful of industrial applications. At the

recent Management Science meeting, Garvin, Crandall, John, and Spellmann reported on a survey of applications in the oil industry. So intense has been the development in this area that it now covers every phase of their work: (1) exploration, (2) production, (3) refining, (4) distribution.

Next to the petroleum industry, the food industry is perhaps the second most active customer. Heinz Company uses it to determine which of a half a dozen plants should ship ketchup to seventy warehouses located in all parts of the country. A major milk producer has been considering it as a means of determining which of its dozen or so milk producing centers should ship canned milk to thousands of warehouses in this country and to foreign markets. Armour and Company use it to determine the most economical mixture of feeds for animals.

In metal processing applications, the president of Argus Cameras reports a savings of \$54,000 per quarter because the price implications of linear programming told them whether to make or to buy a part. The production planning manager of SKF says that the linear programming helped them save 6,000 machine hours each month, or \$100,000 per year over the Gantt Chart Control System used by them prior to 1953.

Important applications are being made in other countries. Thus the Abitibi Power and Paper Company of Canada and the Australian Paper Manufacturing, Ltd. use linear programming methods to minimize paper machine trim losses. It is also



applied to minimizing transportation costs from mills to customers. Recently I received the following communication from a friend visiting in England:

There are some very nice applications around — I have counted a dozen myself, although only two or three have appeared in the literature. In general, firms are keeping their activity quiet for the usual reason that Macy's does not tell Gimbel's.

Elsewhere he says,

A very lovely application was developed by a statistician-economist for planning the operations of one of the divisions of Shell Chemical. It seems to increase the planning profits (based on sales estimates) by about 5%.

There are two interesting fields where linear programming is just beginning to be used:

1. The first was discussed at the 1956 Management Science meeting in Los Angeles by Dr. Pierre Massé, Executive Vice President of the Nationalized Electric Power Industry of France, and Dr. R. Gibrat. They showed how linear programming could be used to decide how much should be invested in new hydroelectric, steam, tidal, and atomic plants.

2. The second was discussed in the October issue of Management Science by R. Kalaba and M. Juncosa in their paper on the application of linear programming to optimal routing of messages in a communications network and to the problem of investment in communication facilities. This work began with the military application but recently research workers in the Bell System have become interested. It seems to offer an



approach to over-all systems studies which are too complicated to handle at present by waiting line theory.

For Air Force applications there is an excellent brochure prepared by the United States Air Force Comptroller<sup>1</sup> that lists a number of typical examples. From this source I have picked two:

The first typical example is a transportation problem. This military example is similar to the Heinz Ketchup example. Assume that Lockbourne Air Force Base at Columbus, Ohio, has been testing a large item of equipment, weighing a ton, for the B-47. It is now desired that this equipment be tried at other bases. Suppose that five each are required by three Air Force bases and three each are required by the two others. To supply these twenty-one items, Columbus and Oklahoma City can ship eight each and Warner-Robins at Macon, Georgia, has five. The items of equipment are to be airlifted to their destinations. The problem is to determine the routing which fulfills the requirements and minimizes the total ton-miles.

There are, of course, many possible routings. The minimum solution can be obtained by one of several methods of linear programming that are now available.

The second typical problem concerns production versus requirements. This military example has its counterpart in many industrial examples.

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<sup>1</sup>"The Application of Linear Programming Techniques to Air Force Problems - A Non-Technical Discussion," Report AFAMA 3-4-52, prepared by Computation Division, Directorate of Management Analysis, DCS/Comptroller, Headquarters U. S. Air Force, December 17, 1954.

Training schools within the Air Force are concerned with the efficient production of graduates. In order to meet a schedule of future requirements, each school must make available the required number of trained men at the right time.

Since each training activity has its own limitations and needs, an efficient method of production for one will not necessarily be efficient for another. Some training activities can readily increase their rates of output while others can do so only within limits and at substantial cost. Where, for example, equipment and personnel must be released when production drops and reinstated as it expands, large fluctuations in the level of production are undesirable. When faced with fluctuating requirements, such an activity will ordinarily overproduce in periods of low requirements, pool the surplus, and use the excess in periods of high requirements. The production pattern can thus be made quite stable. However, a solution may be undesirable if it yields in turn comparatively large monthly surpluses of trained men.

Problems of this nature illustrate the difficulties that arise whenever conflicting objectives are inherent in a problem. Here the desire is to determine a schedule that tries to reduce both large fluctuations in output and large monthly surpluses of men. For these problems, efficient planning means the determination of a middle ground lying between the two extremes.

The optimum production schedule will depend, of course, on the relative importance assigned to the conflicting objectives.